MATH 2050C Lecture on 3/4/2020

Last time ...... we proved some "limit theorems":

$\lim(x_n \pm y_n) = \lim(x_n) \pm \lim(y_n)$; $\lim\left(\frac{x_n}{y_n}\right) = \frac{\lim(x_n)}{\lim(y_n)}$ $\neq 0$

$\lim(x_n y_n) = \lim(x_n)\lim(y_n)$; $x_n \leq y_n \; \forall n$ $(<)$ $\Rightarrow \lim(x_n) \leq \lim(y_n)$

Note: We need to assume $\lim(x_n)$, $\lim(y_n)$ exist !

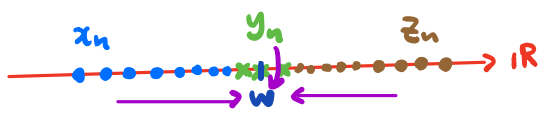
Q: How to prove that limit exists ?

Theorem: (Squeeze Theorem)

Let $(x_n), (y_n), (z_n)$ be sequences of real numbers, s.t.

[or $\forall n \geq K$ for some K]

(1) $x_n \leq y_n \leq z_n \quad \forall n \in \mathbb{N}$



(2) $\lim(x_n) = w = \lim(z_n)$.

Then, (y_n) is convergent and $\lim(y_n) = w$.

Remark: We do NOT need to assume $\lim(y_n)$ exists, this follows as a conclusion of (1) and (2).

Proof: Let $\varepsilon > 0$.

Since $(x_n) \to w$, $\exists K_1 = K_1(\varepsilon) > 0$ s.t. $|x_n - w| < \varepsilon \quad \forall n \geq K_1$

Since $(z_n) \to w$, $\exists K_2 = K_2(\varepsilon) > 0$ s.t. $|z_n - w| < \varepsilon \quad \forall n \geq K_2$

Then, $\forall n \geq K := \max\{K_1, K_2\}$, by (1)

$$-\varepsilon \underset{\forall n \geq K_1}{<} x_n - w \underset{\forall n \in \mathbb{N}}{\leq} y_n - w \leq z_n - w \underset{\forall n \geq K_2}{<} \varepsilon$$

i.e. $|y_n - w| < \varepsilon \quad \forall n \geq K$. $\square$

Example: $\lim\left(\frac{\sin n}{n}\right) = 0$ since $\underset{\downarrow 0}{-\frac{1}{n}} \leq \frac{\sin n}{n} \leq \underset{\downarrow 0}{\frac{1}{n}}$ $\xRightarrow{\text{Squeeze thm}} \left(\frac{\sin n}{n}\right) \to 0$.

Thm: (Ratio test)

Let (x_n) be a sequence in $\mathbb{R}$ s.t.

(1) $x_n > 0 \quad \forall n \in \mathbb{N}$

(2) $\lim \left(\frac{x_{n+1}}{x_n} \right) = L < 1$

Then, $\lim (x_n) = 0$.

Example: Let $(x_n) = \left(\frac{n}{2^n} \right)$. Observe

$$\left(\frac{x_{n+1}}{x_n} \right) = \left(\frac{n+1/2^{n+1}}{n/2^n} \right) = \left(\frac{n+1}{n} \cdot \frac{1}{2} \right) \longrightarrow \frac{1}{2} < 1$$

Ratio test applies $\Rightarrow \lim (x_n) = 0$.

Remark: The thm. is false if $L = 1$. Consider e.g. $(x_n) = (n)$

$$\left(\frac{x_{n+1}}{x_n} \right) = \left(\frac{n+1}{n} \right) \longrightarrow 1 \quad \text{but } (x_n) \text{ divergent}$$

($\because$ unbdd)

Proof: [Idea: Compare (x_n) with a geometric seq. $(b^n \cdot c_0)$ where $c_0 \in \mathbb{R}$ fixed and $0 < b < 1$.]

Assumed: $\lim \left(\frac{x_{n+1}}{x_n} \right) = L$

i.e. $\frac{x_{n+1}}{x_n} \approx L$ for n large

$\Rightarrow x_{n+1} \approx L x_n$ for n large

$\Rightarrow x_{n+2} \approx L^2 x_n \quad \cdots$

Since $L < 1$, we can choose some $r \in \mathbb{R}$

s.t. $L < r < 1$

Take $\varepsilon = r - L > 0$, since $\lim \left(\frac{x_{n+1}}{x_n} \right) = L$,

$\exists\, K = K(\varepsilon) \in \mathbb{N}$ s.t. $\forall n \geq K$,

$$\left| \frac{x_{n+1}}{x_n} - L \right| < \varepsilon = r - L$$

$$\Rightarrow \quad 0 < \frac{x_{n+1}}{x_n} < L + (r - L) = r \quad \forall n \geq K.$$

Therefore, $x_{n+1} < r\, x_n \quad \forall n \geq K$.

More explicitly,

(x_n): $x_1 \quad x_2 \quad \cdots \quad x_{K-1} \quad x_K \quad x_{K+1} \quad x_{K+2} \quad \cdots \quad x_n \quad \cdots$

$(r^{n-K} x_K) \approx x_1 \quad x_2 \quad \cdots \quad x_{K-1} \quad x_K \quad r x_K \quad r^2 x_K \quad \cdots \quad r^{n-K} x_K \quad \cdots$

So, $0 < x_n < r^{n-K} x_K$ and $\lim (r^{n-K} x_K) = 0$ since $0 < r < 1 \Rightarrow$ $\lim (x_n) = 0$ by Squeeze thm.

$\square$

GOAL: When does (x_n) converge / diverge?

Recall: (x_n) convergent $\underset{\not\Leftarrow}{\Rightarrow}$ (x_n) bdd.

equivalently, (x_n) unbdd $\Rightarrow$ (x_n) divergent.

However, (x_n) bdd $\not\Rightarrow$ (x_n) convergent

[E.g. $(x_n) = ((-1)^n)$]

Q: Under what condition(s) does a bdd seq. (x_n) converge?

Monotone Convergence Thm: (x_n) bdd & "monotone" $\Rightarrow$ (x_n) convergent.